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Proof

By Mackey's Thm,

$$\left(\mathrm{Ind}_L^G U\right)_L \cong \bigoplus_{s \in L\backslash G/L} \mathrm{Ind}_{L\cap L^s}^{G} s^{-1}(U)$$

So that $\mathrm{Ind}_L^G U$ has a summand U corr. to $s \in L$ and a sum of modules induced from $L \cap L^s$ for $s \notin L$.

But P is a normal Sylow p-subgp of L with $P \cap P^s = 1$ $\forall s \notin L$ so $P \cap P^s = 1 \in \mathrm{Syl}_p(L \cap L^p)$

So $L \cap L^s$ is a p'-group. But then every $L \cap L^s$-module is Proj^{ve} and so

$$\left(\mathrm{Ind}_L^G U\right)_L = U \oplus Y \quad \text{for } Y \text{ Proj}^{ve}.$$

Def$^{\underline{n}}$

Let V be a KG-module. Then we define the projective cover of V, denoted $P(V)$, to be the minimal proj$^{\text{ve}}$ module with V as a quotient.

If $V = V_1 \oplus \dots \oplus V_n$ is semisimple then

$$P(V) = P(V_1) \oplus P(V_2) \oplus \dots \oplus P(V_n)$$

and more generally $P(V) \cong P(\text{head } V)$

Now, the proj$^{\text{ve}}$ covering map $\pi : P \to M/M'$ lifts to a map $\pi' : P \to M$ whose image is a uniserial submodule of M containing V. As such, $M' \cap \pi'(P) = 0$ and so

$M = M' \oplus \pi'(P)$. Since $\pi'(P) \neq 0$ and M is indecomp., $M' = 0$. $\square$

Lemma

Suppose G has a cyclic normal Sylow p-subgp. Then every PIM for kG is uniserial.

Since $\dim W = 1$, $V \otimes U$ has an irred. submodule $V \otimes W$ with quotient $V \otimes k \cong V$.

Since U is not semisimple, by Lemma 3.23, for a generator x of Q we may choose $0 \neq u \in (1-x)U = \operatorname{rad} U$. If for $0 \neq v \in V$ we have $(1-x)(v \otimes u) \neq 0$ then $\operatorname{rad}(V \otimes U) \neq 0$ and thus $V \otimes U$ is not ss..

As $Q \trianglelefteq G$, V_Q is ss. by Clifford's Thm, and since Q is a p-gp, $(V_Q)^Q = \operatorname{Soc}(V_Q) = V_Q$.

So $xv = v$.

Thus $(1-x)(v \otimes u) = v \otimes u - xv \otimes xu = v \otimes u - v \otimes xu = v \otimes (1-x)u \neq 0$.

Let U_j denote the 1-dim kC-module on which g acts as a^j. Then $U_i \otimes U_j \cong U$ . .

Let U_j denote the 1-dim kL-module on which g acts as a^j. Then $U_{j_1} \otimes U_{j_2} \cong U_{j_1+j_2}$.

Recall that V_2 is the natural kG-module with basis $\{X, Y\}$ where $gX = aX + cY$, $gY = a^{-1}Y$.

So kY is an L-submodule of V_2, iso. to U_{-1} with $(V_2)_L / U_{-1} \cong U_1$. So $(V_2)_L \sim \begin{matrix} U_1 \\ U_{-1} \end{matrix}$

We may take W in this case to be U_{-2}.

Thus if M is indecomp. kL-module w/ head $M \cong U_j$, the radical factors of M are

$$U_j, U_{j-2}, U_{j-4}, \ldots,$$

Proof

Let M be the indecomp. kL-module of dim $p-1$ and head U_{i-1}. Then $M/\mathrm{rad}^i M$ has dim i and head U_{i-1}, thus $M/\mathrm{rad}^i M \cong (V_i)_L$ and head$(\mathrm{rad}^i M)$ is $\mathrm{Soc}(M/\mathrm{rad}^i M) \otimes U_{-2} \cong U_{-i-1}$

But $\dim(\mathrm{rad}^i M) = (p-1)-i$ and so $\mathrm{rad}^i M \cong (V_{p-1-i})_L$ and so we have $\cong U_{(p-1-i)-1}$

$$0 \to (V_{(p-1-i)})_L \to M \to (V_i)_L \to 0$$

which cannot split as M is indecomp. So we are done by Cor. 4.24. $\square$

By ① we have a module $\begin{smallmatrix} V_1 \\ V_{p-2} \end{smallmatrix}$ onto which $P_1 = \mathcal{P}(V_1)$ must surject. So

$$P_1 \sim \begin{matrix} V_1 \\ V_{p-2} \\ X_1 \\ V_1 \end{matrix}$$

for some X_1 (possibly zero)

In particular, $\dim P_1 \geq p$ with equality iff $\mathcal{H}(P_1) \cong V_{p-2}$.
i.e. $X_1 = 0$.

Sim. by ② with $v=2$, $\dim P_{p-1} \geq 2p$ with equality iff $\mathcal{H}(P_{p-1}) \cong V_2$.

Now suppose $1 < i < p-1$, using ① + ② we obtain submodules M_+ and M_- of $\operatorname{rad} P_i$ s.t. $(\operatorname{rad} P_i)/M_+ \cong V_{p+1-i}$ and $(\operatorname{rad} P_i)/M_- \cong V_{p-1-i}$. Letting $M := M_+ \cap M_-$, $(\operatorname{rad} P_i)/M \cong V_{p-1-i} \oplus V_{p+1-i}$

s.t. $(\mathrm{rad}\,P_i)/M_+ = V_{p+1-i}$ and ...

$$P_i \sim \begin{array}{c} V_i \\ V_{p-1-i} \oplus V_{p+1-i} \\ X_i \\ V_i \end{array}$$ for some (maybe zero) X_i.

Again, $\dim P_i \geqslant 2p$ with eq. $\iff X_i = 0$. Moreover,

$$p(p^2-1) = \dim kG = \sum_{i=1}^{p} \dim V_i \dim P_i$$
$$\geqslant p + 2p\sum_{i=2}^{p-1} i + (p-1)2p + p^2$$
$$= p^3 - p = p(p^2-1)$$

Fix the notation $A = A_1 \oplus \cdots \oplus A_n$. Take an A-module M. If $A_i M = M$ and $A_j M = 0$ $\forall i \neq j$ we say M lies in the block A_i.

We say M lies in the block A_i.

Take idecomp. $1 = e_1 + e_2 + \dots + e_n$ for e_i the identity of A_i. If M lies in A_i, then e_i acts as the identity on M and $e_j M = 0 \ \forall j \neq i$.

If M lies in A_i, so do all submodules and quotients of M. Further, direct sums of modules in A_i also lie in A_i.

If M_i lies in A_i and M_j lies in A_j for $i \neq j$, then $\mathrm{Hom}_A(M_i, M_j) = 0$ since for any $\varphi : M_i \to M_j$ we have that $e_i \varphi(M_i)$ is the identity but $e_i M_j = 0$.

$\underline{\text{Th}^{\underline{m}}}$

Let M be an A-module. Then M has a unique decomp.

$$M = M_1 \oplus \dots \oplus M_n$$

Where M_i lies in the block A_i.